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ON THE DRAG OF BODIES OF REVOLUTION AT TRANSONIC SPEEDS

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Dependence of the pattern of variation of gas parameters on the deviation of the oncoming stream velocity from sonic is established as the result of investigation of flow at great distances from bodies of revolution. This dependence makes it possible to determine the law of drag variation at transonic speeds, which is confirmed by calculations presented here.

The weak effect of the oncoming stream velocity on the deviation of parameters at the body upstream of the compression shock from their values at sonic speed at infinity is a property of transonic flows, known as the law of stabilization. It was discovered experimentally and expounded in [1] for plane flows. The relation of the stabilization law to the pattern of the stream at great distances upstream of a compression shock was established in [2, 3]. In the first of these the assumption is made that the drag is weakly dependent also on the velocity at infinity, which is not supported by experimental data. The latter reveal a rapid motion of the compression shock toward the body trailing edge, when the velocity of the oncoming stream approaches the speed of sound. For constant parameters upstream of the shock the drag is affected not only by the motion of the shock itself, but also by parameters downstream of it. For the determination of the dependence of the drag of a body on the oncoming stream velocity, it is, consequently, necessary to investigate the flow downstream of the shock.

1. Let us briefly state the properties of sonic flows at great distances from a body of revolution, which will be required subsequently. They were investigated in [4 – 11] and provide a fairly complete picture of the flow as a whole. In particular, they clarify the nature of incipient formation of drag of a body at sonic velocity.

Since investigations [4 – 11] imply that at great distances from a body the compression shock intensity is low, hence there exists a velocity potential which can be represented by an infinite asymptotic series. Below we shall need only three of its terms which are of the form

$$\Phi = a_* x + \sum_{k=1}^{3} y^{-2k/7} \varphi_k(\xi), \qquad \xi = xy^{-4/7}(\varkappa + 1)^{-1/3} \tag{1.1}$$

where a_* is the critical speed of sound, $\varkappa$ is the exponent of the Poisson adiabate, and x and y are axes of a cylindrical coordinate system. Functions $\varphi_k(\xi)$ are determined with an accuracy to within the constant coefficients, with $\varphi_2(\xi)$ nonzero only downstream of the compression shock. In conformity with (1.1) the shock front is defined by the equality

$$x = (\varkappa + 1)^{1/3} y^{4/7} \xi_S (1 + c_2 y^{-2/7} + c_3 y^{-4/7}) \tag{1.2}$$

in which ξ_S, c_2 and c_3 are constants.

Let us consider a closed reference surface surrounding the body and consisting of a cylinder $y = R$ and two planes $x = \pm(\varkappa + 1)^{1/3} \xi_S R^{4/7}$ $(R \gg 1)$. The rate of gas flow through it is calculated by the asymptotic expansion (1.1) and is equal $A + O(R^{-2/7})$, where A is proportional to c_2 [9]. Since the closed body cannot be a source of mass, formula (1.1) is inapplicable on some of the regions of the flow field at great distances (from the body). Below we denote by I the region where (1.1) is valid. To avoid physical inconsistency of the flow pattern, it is necessary to consider also the region of the vortex trail (region II) lying along the x-axis downstream of the body. The principal term of the expansion of solution for that region with $x \to \infty$ represents a shear flow

$$v_y = 0, \quad v_x = a_* u(\psi), \quad S = S_* + \frac{\varkappa}{\varkappa - 1} \ln\left[\frac{\varkappa + 1}{2}\left(1 - \frac{\varkappa - 1}{\varkappa + 1} u^2\right)\right] \tag{1.3}$$

completely determined by the entropy change $S(\psi) - S_*$ of particles which takes place at intersection with the compression shock, and is considered specified. Subsequent terms of the expansion of solution in the trail which depend on x were derived in [10] in xy-coordinates and not in Mises coordinates x, ψ which are more suitable for our purpose. The deficiency of flow through the trail region produced by the entropy distribution in it compensates the excess of flow through the external region and results in the drag defined by formula

$$F_x = 2\pi\rho_* a_*^2 \int_0^\infty u^{-1}(1 - u^2)(1 - \mu^2 u^2)^{-\varkappa/(\varkappa-1)} d\Psi, \quad \mu^2 = \frac{\varkappa - 1}{\varkappa + 1} \tag{1.4}$$

$$\Psi = \int_0^\psi \left(\frac{\varkappa + 1}{2}\right)^{-1/(\varkappa-1)} e^{S-S_*} d\psi$$

where ρ_* is the critical density.

2. Let us pass to the investigation of flows which at very great distances from the body differ only slightly from flows at sonic velocity and introduce a certain small parameter ε to characterize such difference. At great distances from a body of revolution outside the trail (region I) we represent the flow velocity potential in the form

$$\Phi = a_* x + \varphi(x, y) + \varepsilon\chi(x, y) \tag{2.1}$$

where $\varphi(x, y)$ denotes the potential of the sonic stream perturbation, as defined by (1.1), and $\chi(x,y)$ denotes its perturbation related to the change of boundary conditions at infinity. We seek the principal term of expansion of $\chi(x,y)$ of the form

$$\chi = y^{-2m/7} \chi_m(\xi), \quad m < 1 \tag{2.2}$$

We assume that the constant m which defines $\chi(x, y)$ is known. Let us consider the principal term of the complete perturbation of potential

$$y^{-2/7}\varphi_1(\xi) + \varepsilon y^{-2m/7}\chi_m(\xi) \qquad (2.3)$$

and assume that the difference of boundary conditions at infinity from sonic are related only to the divergence of the oncoming stream velocity M_∞ from unity. Then the same form (2.3) remains valid for various M_∞, with only parameter ε dependent on M_∞. To determine the character of that dependence we carry out the transonic similarity transformation

$$x \to x / x_0, \quad y \to y|1 - M_\infty|^{1/2} / x_0$$

stipulating the conservation of ξ. The ratio of terms of the sum (2.3) then yields the required dependence

$$\varepsilon = (1 - M_\infty)^{-(m-1)/3} \qquad (2.4)$$

The condition $m < 1$ appears in (2.2) which defines the principal term of the expansion of potential $\chi(x, y)$. It implies the existence of distances at which the second term in (2.3) becomes comparable to the first, and where expansion in terms of the small parameter ε is no longer valid. The order of magnitude of such distances, which is determined by the comparison of terms in (2.3) at constant ξ, yields estimates

$$y = O[|1 - M_\infty|^{-7/6}], \qquad |x| = O[(1 - M_\infty)^{-2/3}]$$

for the external boundaries of the region of applicability of all expansions derived below. These estimates show that in the considered region flow parameters do not greatly differ from their corresponding values in a sonic stream, hence, for example, for $M_\infty < 1$ the end of a local supersonic zone and the place of the compression shock onset do not belong to that region.

3. Thus, the determination of m is the principal task of the problem. To achieve it we write the equation which must be satisfied by function $\chi_m(\xi)$ itself [2, 12]

$$\left(\frac{d\varphi_2}{d\xi} - \frac{16}{49}\xi^2\right)\frac{d^2\chi_m}{d\xi^2} + \left[\frac{d^2\varphi_1}{d\xi^2} - \frac{16}{49}(m+1)\xi\right]\frac{d\chi_m}{d\xi} - \frac{4}{49}m^2\chi_m = 0 \qquad (3.1)$$

The limit characteristic $\xi = \text{const}$ along which $d\varphi_1 / d\xi = {}^{16}/_{49}\,\xi^2$ is a singular line for this equation. The highest value of m for which χ_m is analytic upstream of the compression shock and satisfies the symmetry condition at passing over the x-axis for $x < 0$ was found in [2, 12] to be four.

In accordance with the above the difference between the flow parameters at $M_\infty \neq 1$ and those at sonic velocity is proportional to $(1 - M_\infty)^{5/3}$, and this provides the theoretical basis for the experimentally established stabilization of parameters ahead of the compression shock in the flow past a body in the transonic velocity range [2, 3]. The other experimental result which shows the rapid motion of the shock toward the body trailing edge with increasing M_∞ can only be explained by the existence in the region downstream of the shock of solutions of Eq. (3.1) for $m > -4$.

At the shock front whose shape in accordance with (1.2), (2.1) and (2.2) is defined by equation (3.2)

$$x = (\varkappa + 1)^{1/3}\xi_S y^{4/7}(1 + c_2 y^{-2/7} + c_3 y^{-4/7} + \ldots + \varepsilon D_m y^{2(1-m)/7})$$

the condition of continuity of the potential and the equation of the shock polar must be satisfied. For $\xi = \xi_S$ the transformation of these in conformity with (2.3) yields for Eq. (3.1) the Cauchy data.

Solution of the obtained Cauchy problem for Eq. (3.1) satisfies for $\xi > \xi_S$ the condition of absence of sources along the x-axis only for $m = 2$ [9, 10]. Since $m = 2$ is outside the considered range of variation $-4 < m < 1$, the perturbed flow for $\xi \gg \xi_S$ is defined by the two linearly independent solutions of Eq. (3.1). The analysis of their behavior for $\xi \to \infty$ yields the estimate

$$\chi_m(\xi) \to \xi^{-m/2}(A_{m,1} + A_{m,2}\ln\xi) \qquad (3.3)$$

where $A_{m,1}$ and $A_{m,2}$ are constants proportional to D_m.

For the subsequent analysis it is necessary to calculate the increments v_x' and v_y' induced in velocity components by the potential $y^{-2m/7}\chi_m(\xi)$. In the neighborhood of the x-axis we obtain in accordance with formula (3.3) the estimates

$$v_x' = O(x^{-1-m/2}\ln\xi),\ v_y' \to B_m x^{-m/2}y^{-1},\ B_m = -{}^4/_7 A_{m,2}(\varkappa+1)^{m/6} \qquad (3.4)$$

Formulas (3.4) show that a perturbation of sonic conditions at infinity results in infinitely high transverse velocities near the x-axis. This is because expansion (2.1) does not hold in the region whose cross section is of the order of ε (region III) adjacent to the x-axis. This phenomenon does not occur when $M_\infty = 1$.

4. Before eliminating the singularity for $y \to 0$, let us consider the flow in the region of the vortex trail (region II) in which, as shown in Sect. 2, the velocity potential is absent. We take the coordinates x and Ψ as the independent variables, and the increments v_x' and v_y' of velocity, S' of entropy, and ψ' of the stream function as the unknown functions. All of these are proportional to the small parameter ε. The flow in region II is defined by the Crocco equations of continuity, that of adiabacity, and by the equations for ψ' linearized with respect to solution (1.3). These together with the three boundary conditions for $\Psi \to \infty$, derived with the use of the principle of asymptotic joining with the solution in region I for considerable x [13]

$$S' = v_x' = 0, \qquad y v_y' = B_m x^{-m/2} \qquad (4.1)$$

completely define the stream in the considered region.

In accordance with (4.1) we seek the solution of these equations of the form

$$v_x' = x^{1-m/2}f_m(\Psi),\ \ v_y' = x^{-m/2}g_m(\Psi) \qquad (4.2)$$

$$S' = x^{1-m/2}\sigma_m(\Psi),\ \ \psi' = x^{1-m/2}\pi_m(\Psi)$$

The resulting system of linear equations is readily integrated

$$f_m = \frac{2}{m-2}B_m(1-\mu^2u^2)^{1/(\varkappa-1)}u\frac{du}{d\Psi},\quad g_m = B_m u y^{-1} \qquad (4.3)$$

$$\sigma_m = \frac{2}{2-m}B_m(1-\mu^2u^2)^{1/(\varkappa-1)}u\frac{dS}{d\Psi}$$

$$\pi_m = \frac{2}{m-2}B_m\left(\frac{\varkappa+1}{2}\right)^{1/(\varkappa-1)}e^{S_*-S}u(1-\mu^2u^2)^{1/(\varkappa-1)}$$

It follows from solution (4.3) that the boundary conditions (4.1) are in fact satisfied, hence for $\Psi \to \infty$ the derivatives $du/d\Psi \to 0$ and $dS/d\Psi \to 0$ [10]. Furthermore the additional flow $\psi' = 2\,(m-2)^{-1}B_m x^{1-m/2}$ for $\Psi \to \infty$ and for $\Psi = 0$ is nonzero. The last conclusion clearly shows that the vertical velocity $v_y = \varepsilon v_y'$ tends to infinity as y^{-1} when approaching the x-axis. This result makes it necessary to introduce the subregion of the vortex trail (region III) which lies in the immediate vicinity of the x-axis. Its transverse dimension, as well as the transverse velocity of particles are of the order of $\varepsilon^{1/2}$, while the horizontal velocity $v_x = O\,(1)$. The principal term of the expansion of parameters of the stream in that region in terms of ε yields a uniform flow at

$$v_x = a_* u\,(0), \quad v_y = 0 \tag{4.4}$$

The stream function in region III varies from $\psi'\,(x,\,0)$ to zero. This shows that the total deficiency of flow through the two regions II and III is equal $2\pi\varepsilon\psi'\,(x,\,\infty)$. Computation by solution (4.2) – (4.4) of the excess of total value of the x-component of the momentum flux carried through the plane $x =$ const which intersects regions II and III shows it to be zero, although for parts of that plane which intersect each of these regions it is nonzero and proportional to $x^{1-m/2}$.

5. The derived asymptotic picture of flow satisfies the conditions at both the compression shock and the axis of symmetry $y = 0$. It remains to determine the value of m for which it is valid.

For this let us consider the following terms of expansion of the additional potential defined by

$$\chi = \sum_{k=0} y^{-2(m+k)/7}\chi_{m+k}\,(\xi) \tag{5.1}$$

Each function χ_{m+k} satisfies the linear differential equation whose left-hand part is the same as in (3.1), if in the latter $m + k$ is substituted for m and the right-hand part is the sum

$$-\sum_{p=1}^{k} \frac{d}{d\xi}\left(\frac{d\varphi_{p+1}}{d\xi}\,\frac{d\chi_{m+k-p}}{d\xi}\right) + \alpha\,(\varphi_p,\ \chi_{m+k-p})$$

In accordance with (2.1) and (5.1) the shock is specified by the equality

$$x = (\varkappa + 1)^{1/4}\,\xi_S y^{4/7}\left\{1 + c_2 y^{-2/7} + c_3 y^{-4/7} + \ldots + \varepsilon \sum_{k=0} D_{m+k} y^{-2(m+k-1)/7}\right\} \tag{5.2}$$

Along the line $\xi = \xi_S$ functions $\chi_{m+k}\,(\xi)$ conform to the Cauchy conditions which are not adduced here owing to the unwieldiness of obtained formulas. It should be noted, however, that the Cauchy problem for each function χ_{m+k} depends on the arbitrary parameter D_{m+k} appearing in (5.2) which defines the compression shock front. In region I solution (5.1) generates related expansions in regions II and III, whose every term is defined by formulas (4.2) and (4.3) when $m + k$ is substituted in these for m. A particular case occurs for $m + k = 2$. This case is considered in detail below.

Let us determine the entropy gained by particles at the intersection with the compression shock using for this the solution for region I. It is defined by

$$S = b_1 y^{-18/7} + \varepsilon b_2 y^{-2(m+8)/7} + \ldots, \qquad (b_1 \text{ and } b_2 \text{ are constants})$$

In this formula the term proportional to ε is independent of x. Hence for joining the entropies in region I and II the expansion in region II must also contain a term which is independent of x. It follows from (4.2) that this is only possible for $m + k = 2$. Since k is a positive integer, m can have only negative values $0, -1, -2, \ldots$, of which the highest is zero.

The derived value $m = 0$ solves the formulated here problem of determination of the principal term of the additional potential $\chi(x, y)$ downstream of the compression shock for $M_\infty \neq 1$. The difference between stream parameters in this region from those obtaining for $M_\infty = 1$ is in accordance with (2.4) proportional to $(1 - M_\infty)^{1/3}$.

6. The irregular dependence on m is the consequence of the unsuitability of formulas (4.3) for the calculation of f_{m+k}, π_{m+k} and σ_{m+k} for $m + k = 2$, which is related to the change of the kind of dependence of longitudinal velocity, entropy, and stream function perturbations on x Formulas (4.2) show that for $m + k = 2$ the sought functions are independent of x It is then necessary to consider instead of (4.2) a more complex form of solution which is the sum of two terms, the first of which is independent of x and the second is proportional to $\ln x$. Existence of the second term makes it possible to obtain a solution with $g_2(\Psi) \not\equiv 0$. If, however, in the stream $g_2(\Psi) \equiv 0$, the sought solution contains the first term only which is independent of x.

The system of equations for second terms is homogeneous. For $\Psi \to \infty$ its solution must satisfy conditions (4.1) and, if in these $B_2 = 0$, the solution of the system is trivial. If $m = 0$, the quantity B_2 can be determined, since then the equation for χ_2 has the first integral

$$\left(\frac{d\varphi_1}{d\xi} - \frac{16}{49}\xi^2\right)\frac{d\chi_2}{d\xi} - \frac{16}{49}\xi\chi_2 + \frac{d\varphi_2}{d\xi}\frac{d\chi_1}{d\xi} + \frac{d\varphi_3}{d\xi}\frac{d\chi_0}{d\xi} = E$$

where E is a constant which for $\xi = \xi_S$ is determined by Cauchy relationships. It vanishes for any arbitrary D_0, D_1 and D_2, which implies $B_2 = 0$.

The term of expansions of stream parameters in region II which depend on χ_2 can be determined, if the additional entropy change $\varepsilon\sigma_2(\Psi)$ generated at the passing of particles through the distorted compression shock at great distances from a body of revolution is known. These terms are

$$f_2 = \frac{\varkappa + 1}{2\varkappa}\,\frac{1 - \mu^2 u^2}{u}\,\sigma_2, \qquad g_2 = 0$$

$$\pi_2 = \frac{1}{\varkappa}\int_0^\Psi \frac{1 + \mu^2 u^2}{u}(1 - \mu^2 u^2)^{-\varkappa/(\varkappa-1)}\sigma_2\, d\Psi$$

Function $\sigma_2(\Psi)$ tends to zero as $\Psi^{-8/7}$ when $\Psi \to \infty$, hence $\pi_2(\infty)$ is bounded. Depending on its value, parameter D_2 in formula (5.2) is specified so as to satisfy the equality

$$(2\pi)^{-1}Q_2' = {}^4/_7\,(\varkappa + 1)^{1/3}\lim_{\xi\to\infty}\xi\chi_2 = \pi_2(\infty)$$

which defines the continuity of flow at the passage from region I to II.

The considered flow has an additional momentum flux which increases the drag (1.4) by $\varepsilon F_x'$. Its magnitude is completely determined by the change of entropy in the compression shock

$$F_x' = 4\pi\varkappa^{-1}\rho_* a_*^2\int_0^\infty \sigma_2 u^{-1}(1 - \mu^2 u^2)^{-\varkappa/(\varkappa-1)}\, d\Psi \tag{6.1}$$

7. In some cases of flow past bodies of revolution the integral (6.1) may be zero. Let us assume that $\sigma_2(\Psi) \equiv 0$. Then in (2.1) $m = -1$, $\varepsilon = (1 - M_\infty)^{2/3}$ and $F_x' = a\varepsilon + d\varepsilon\ln\varepsilon$. Constants a and d may be of different order. This particular situation has occurred in the course of computation of flows past two bodies of revolution with a smooth tail section. The computation was carried out by the difference relaxation

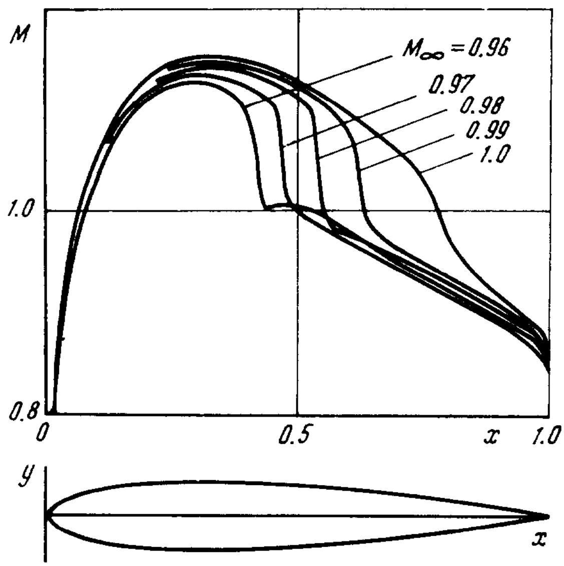

Fig. 1

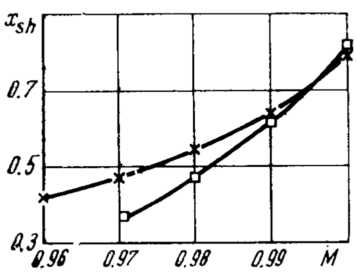

Fig. 2

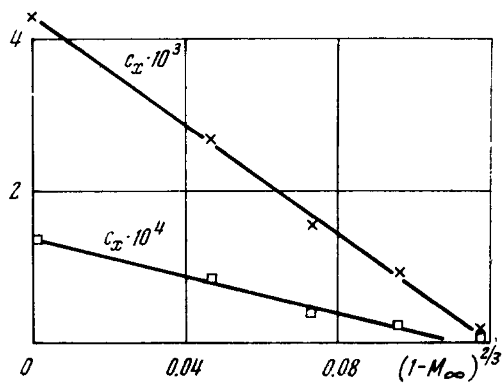

Fig. 3

method similar to that expounded in [14] but applied to bodies of revolution.

Distribution of local Mach numbers M along one of these bodies of revolution for several velocities of the oncoming stream is shown in Fig. 1. The meridian cross section of the body had the form of a Chaplygin profile. It is seen that the position of the compression shock changes much quicker than the increase of velocities at points of the body ahead of the shock. The analysis of their variation in [3] yielded a linear dependence on $(1 - M_\infty)^{5/3}$, which is a numerical confirmation of the stabilization law.

The drag of a body of revolution can be computed by the distribution of parameters along it by the simple integration of the pressure coefficient. The change of the shock position along the body or in its immediate vicinity obviously provides the basic contribution to the variation of the integral. Coordinates $x = x_{sh}$ of the compression shock for the two considered bodies of revolution at distance $y = 0.1$ from these are shown in Fig. 2 for various M_∞. These coordinates were determined by computing points of the maximum velocity gradient. The analysis of these coordinates by formula

$$\lg [x_{sh}(1) - x_{sh}(M_\infty)] = K + n \lg (1 - M_\infty)$$

shows that in both cases $n = 0.66 \approx {}^2/_3$. Curves of the drag coefficient plotted in Fig. 3 which are in complete agreement with the obtained value of n, show the linear dependence [of the drag coefficient] on $(1 - M_\infty)^{5/3}$. This is in good agreement with the results of the above analysis of the asymptotic properties of axisymmetric transonic flows at $M_\infty \neq 1$.

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